

# FROM SELF-SIMILAR STRUCTURES TO SELF-SIMILAR GROUPS

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**ABSTRACT.** We explore the relationship between limit spaces of contracting self-similar groups and self-similar structures. We give the condition on a contracting group such that its limit space admits a self-similar structure, and also the condition such that this self-similar structure is p.c.f. We then give the necessary and sufficient condition on a p.c.f. self-similar structure such that there exists a contracting group whose limit space has an isomorphic self-similar structure; in this case, we provide a construction that produces such a contracting group. Finally, we illustrate our results with several examples.

## 1. INTRODUCTION

The theory of self-similar groups developed as a part of geometric group theory in the last decades. In this theory, many exotic groups (such as groups of intermediate growth, non-elementary amenable groups, and infinite finitely generated torsion groups) could be easily described by their actions on a rooted tree [Gri80, Gri84, GS83, BGŠ03]. More recently, a close relationship between the theory of self-similar groups and fractals has been discovered and studied [BGN03, Nek05, NT08]. This survey aims to clarify this relationship by closely examining the correspondence between self-similar groups and self-similar structures on fractals.

The standard reference on the theory of self-similar groups is [Nek05]. A *self-similar group* is an automorphism group acting on the rooted tree in a recursive manner. Every *contracting* self-similar group  $G$  (see Definition 2.4) induces an *asymptotic equivalence relation* on the *boundary* of the rooted tree (i.e. the space of left-finite words), and the quotient space of the boundary of the rooted tree by this equivalence relation is called the *limit space*  $\mathcal{J}_G$  of the self-similar group.  $G$  does not act on  $\mathcal{J}_G$ , but rather contains information about the adjacency of cells of  $\mathcal{J}_G$  and describes its fractal-like properties.

Of these properties, we are most interested in the self-similarity of the limit space; to examine this, we employ the notion of *self-similar structures*, which is fundamental in analysis on fractals. Self-similar structures have been extensively studied in [Kig01]. A self-similar structure is a finite set of injections from a compact space  $K$  to itself, such that  $K$  is covered by the union of the images. By repeatedly applying these injections, each point in  $K$  can then be given some *addresses* in the *code space*, which can be identified with the boundary of a rooted tree; in this way,  $K$  can also be viewed as a quotient space of the boundary of the rooted tree. Many well-known fractals can be given a self-similar structure.

To capture the self-similarity of the limit space of a contracting self-similar group, we only consider the self-similar structure on the limit space where the two quotient maps mentioned above are the same. Such a self-similar structure may not exist (see Example 6.1), but if it does, then the action of the group can aid the development of analytic properties

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of the limit space (see [NT08]). A natural question arises: Under what conditions on the self-similar group does this self-similar structure exist? We attempt to give an answer in Theorem 3.5.

In the second half of the paper, we focus on *post-critically finite* (p.c.f.) self-similar structures on limit spaces. A p.c.f. fractal is one where the cells of the fractal only intersect at finitely many points, and these points have finitely many addresses. This is a broad class of fractals on which the methods to develop a Laplacian, as well as an analogous Gaussian process, are known. For more details on p.c.f. self-similar structures and their analytic properties, see [Kig93, Kig01, Str06]. We investigate the conditions for which the self-similar structure on a limit space is p.c.f. (see Theorem 4.7). We also attempt to answer the inverse question: Given a p.c.f. self-similar structure, can we find a contracting self-similar group whose limit space has an isomorphic self-similar structure? In Section 5, we shall identify the condition on the self-similar structure (Theorem 5.9) such that the answer is affirmative, and attempt to directly construct a contracting self-similar group in this case.

In practice, we see that certain fractals, equipped with any self-similar structure, cannot arise as the limit space of a contracting action. This includes both non-p.c.f. fractals (such as the diamond fractal) and p.c.f. fractals (such as the Linström snowflake). For some fractals (including the Sierpiński gasket and the pentakun), a contracting group can be found only for some (and not all) self-similar structures on the space. Also, we shall exhibit that two contracting groups that are not isomorphic can have the same limit space with the same self-similar structure in Example 6.4.

This paper is organized as follows. We begin with a brief review of the basic definitions of self-similar groups and limit spaces in Section 2. Section 3 makes precise the notion of the self-similar structure on a limit space, and gives the condition on the contracting group that ensures the existence of this induced self-similar structure. Building on the work in [BN03, NT08], we discuss p.c.f. self-similar structures on limit spaces in Section 4, and in particular give the condition for the self-similar structure on a limit space to be p.c.f. In Section 5, we address the inverse problem: We detail a construction that, given a p.c.f. self-similar structure with a certain necessary condition, produces a contracting self-similar group whose limit space has an isomorphic self-similar structure. Finally, Section 6 is a compilation of examples illustrating the findings of this paper.

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## 2. PRELIMINARIES

We begin by reviewing the basic definitions in the theory of self-similar groups. For more details, see [BGN03, Nek05, NT08].

Let  $X$  be a finite set, called the *alphabet*. We denote by  $X^n$  all *finite words*  $w = x_n \dots x_2 x_1$  of length  $n$  over  $X$ , where  $x_i \in X$ . The length of the word  $w$  is denoted by  $|w|$ . The set of all finite words, including the empty word  $\emptyset$ , is denoted by  $X^* = \bigcup_{n=0}^{\infty} X^n$ . The set  $X^*$  has a natural structure of a rooted tree with the root  $\emptyset$ , where a word  $w \in X^*$  is connected by an edge to each of the words of the form  $xw$ , where  $x \in X$ .

Consider the set  $X^{-\omega} = \{\dots x_2 x_1 : x_i \in X\}$  of all *left-infinite words*. There is a natural topology on the disjoint union  $X^* \sqcup X^{-\omega}$  given by the basis consisting of open sets of the form  $X^* w \sqcup X^{-\omega} w$ , where  $X^* w$  and  $X^{-\omega} w$  denote the sets of words ending by the finite word  $w$ . In this topology,  $X^{-\omega}$  is homeomorphic to the countable product of the discrete set  $X$ , and therefore to the Cantor set.

For all  $w \in X^*$ , we identify  $w$  with the map  $\tilde{w} : X^{-\omega} \rightarrow X^{-\omega}$ , defined by  $\tilde{w}(\dots x_2 x_1) = \dots x_2 x_1 w$ . We also define the shift map  $\sigma : X^{-\omega} \rightarrow X^{-\omega}$  by  $\sigma(\dots x_2 x_1) = \dots x_3 x_2$ .

An *automorphism* of the rooted tree  $X^*$  is a permutation of  $X^*$  that fixes  $\emptyset$  and preserves adjacency of the vertices. The group of all automorphisms of  $X^*$  is denoted by  $\text{Aut } X^*$ . We shall denote the identity automorphism by 1. Every automorphism  $g \in \text{Aut } X^*$  preserves the *levels*, so that  $|g(w)| = |w|$  for every  $w \in X^*$ .

Let  $g \in \text{Aut } X^*$ . If we identify the first level  $X^1$  of the rooted tree with  $X$ , then the restriction of  $g$  to  $X^1$  is a permutation of  $X$ , which will be called the *root permutation* of  $g$  and denoted  $\sigma_g$ . For every  $x \in X$ , if we identify both the subtrees  $gX^*$  and  $\sigma_g(x)X^*$ , then the restriction of  $g$  to  $xX^*$  is another automorphism of  $X^*$ , called the *restriction* of  $g$  at  $x$  and denoted  $g|_x$ . Then we can write  $g(xw) = \sigma_g(x)g|_x(w)$  for all  $x \in X$  and  $w \in X^*$ .

More generally, for each  $v \in X^*$  we identify the subtrees  $vX^*$  and  $g(v)X^*$  and write  $g(vw) = g(v)g|_v(w)$ . (We define  $g|_\emptyset = g$ .) Then we have the basic identities

$$\begin{aligned} g|_{v_1 v_2} &= g|_{v_1} |_{v_2}, \\ (g_1 g_2)|_v &= g_1|_{g_2(v)} g_2|_v. \end{aligned}$$

This gives us

$$g_1 g_2(xw) = g_1 g_2(x) g_1|_{g_2(x)} g_2|_x(w).$$

We shall also use the “wreath recursion” notation to express this. If  $X = \{0, \dots, k-1\}$ , then

$$\begin{aligned} g &= \sigma_g(g|_0, g|_1, \dots, g|_{k-1}), \\ g_1 g_2 &= \sigma_{g_1} \sigma_{g_2}(g_1|_{\sigma_{g_2}(0)} g_2|_0, \dots, g_1|_{\sigma_{g_2}(k-1)} g_2|_{k-1}). \end{aligned}$$

**Definition 2.1.** A faithful action of a group  $G$  on the rooted tree  $X^*$  is said to be *self-similar*, or *state-closed*, if for every  $g \in G$  and every  $x \in X$  there exist  $h \in G$  and  $y \in X$  such that  $g(xw) = yh(w)$  for every  $w \in X^*$ .

We will denote such an action as  $(G, X)$ . If  $g(xw) = yh(w)$ , then obviously  $y = g(x)$  and  $h = g|_x$ . We will also write  $g \cdot x = y \cdot h$ . Given a faithful action of  $G$  on  $X^*$ , there is a natural isomorphism between  $G$  and a subgroup of  $\text{Aut } X^*$ , with which it will be identified. Thus, we will also use the terms *self-similar subgroup* of  $\text{Aut } X^*$  and *self-similar automorphism group* to describe a self-similar action.

A set  $S \subset \text{Aut } X^*$  of automorphisms is said to be *state-closed* if the restriction of every  $g \in S$  to every  $x \in X$  is also in  $S$ . If every element of  $S$  has finite order, then the group  $G = \langle S \rangle$  is self-similar.

The notion of a self-similar action is closely related to that of an automaton [ECH<sup>+</sup>92, Wol02].

**Definition 2.2.** An *automaton*  $A$  over the alphabet  $X$  is a set of *internal states*, also denoted  $A$ , together with a map  $\tau : A \times X \rightarrow X \times A$ .

An automaton is *finite* if and only if its set of internal states is finite. If  $\tau(q, x) = (y, p)$ , we will also write formally  $q \cdot x = y \cdot p$ . For every state  $q \in A$ , we can define the *action of the state*  $q$  on all finite words  $w = x_n \dots x_2 x_1$ , by processing the letters one by one: it reads the first letter  $x$  of  $w$ , outputs the letter  $p = q(x)$ , goes to the state  $y = q|_x$  and goes on to read the next letter. At the end it will give as output some word  $q(w)$  where  $|q(w)| = |w|$ , and stop at some state of  $A$ .

An automaton  $A$  is often represented by its *Moore diagram*, which is a directed graph with the set of vertices  $A$ , in which for every  $x \in X$  and every  $q \in A$ , there is an arrow from  $q$  to  $q|_x$  labeled  $(x, q(x))$ . Then for  $q \in A$  and  $w = x_n \dots x_2 x_1 \in X^*$ , the image

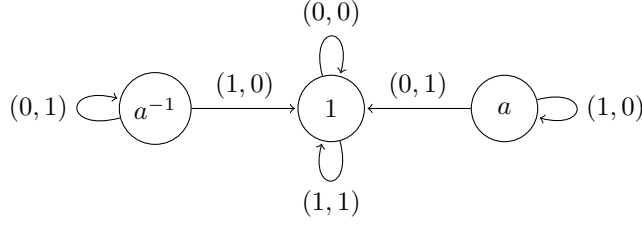


FIGURE 2.1. Nucleus of the binary adding machine

$q(w)$  under the action of the state  $q$  can be found by finding a path in the Moore diagram which starts at  $q$  with consecutive labels of the form  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ ; then  $q(w) = y_n \dots y_2 y_1$ .

The relationship between self-similar actions and automata is illustrated below.

**Definition 2.3.** Let  $(G, X)$  be a self-similar action. An automaton  $A$  is said to be the *complete automaton* of  $G$  if its set of internal states is  $G$  and the action of the states coincides with the action of  $G$ .

It is routine to prove that there is a one-to-one correspondence between the self-similar actions and their complete automata. Henceforth, we will identify a self-similar action with its complete automaton, and denote a state of the automaton by its corresponding group element  $g$ .

**Definition 2.4.** A self-similar action is said to be *contracting* if there exists a finite set  $\mathcal{N} \subset G$  such that for every  $g \in G$ , there exists  $k \in \mathbb{N}$  such that  $g|_w \in \mathcal{N}$  whenever  $|w| \geq k$ . The smallest such  $\mathcal{N}$  is called the *nucleus* of  $G$ .

Figure 2.1 shows the Moore diagram of the nucleus of the binary adding machine, one of the simplest contracting self-similar groups. For more on this group, see Example 6.1.

Contracting self-similar actions have an associated topological space, which we describe below.

**Definition 2.5.** Let  $(G, X)$  be a contracting action. Two left-infinite words  $\dots x_2 x_1, \dots y_2 y_1 \in X^{-\omega}$  are said to be *asymptotically equivalent* if there exists a sequence  $\{g_k\}_{k \geq 1}$  of group elements, taking only finitely many values, such that  $g(x_k \dots x_2 x_1) = y_k \dots y_2 y_1$  for every  $k \geq 1$ . The quotient space of  $X^{-\omega}$  by the asymptotic equivalence relation is called the *limit space* of the action and is denoted  $\mathcal{J}_G$ ; we denote the quotient map by  $p : X^{-\omega} \rightarrow \mathcal{J}_G$ .

We shall use the following more useful characterization of asymptotic equivalence.

**Theorem 2.6** ([Nek05] Theorem 3.6.3). *Let  $(G, X)$  be a contracting action. Two left-infinite words  $\dots x_2 x_1, \dots y_2 y_1 \in X^{-\omega}$  are asymptotically equivalent if and only if there exists a left-infinite path  $\dots e_2 e_1$  in the Moore diagram of the nucleus such that the edge  $e_n$  is labeled by  $(x_n, y_n)$ .*

*The topological space  $\mathcal{J}_G$  is compact, metrizable and has topological dimension less than the size of the nucleus.*

Figure 2.2 shows the Moore diagram of the nucleus of the 3-peg Hanoi Towers Group, which is a well-known contracting self-similar group. Its limit space has been shown in [GŠ06, GŠ08] to be the Sierpiński gasket. For more about this group, see Example 6.2

We now mention a property of the quotient map  $p$ , and the definition of the induced shift map  $s$ .

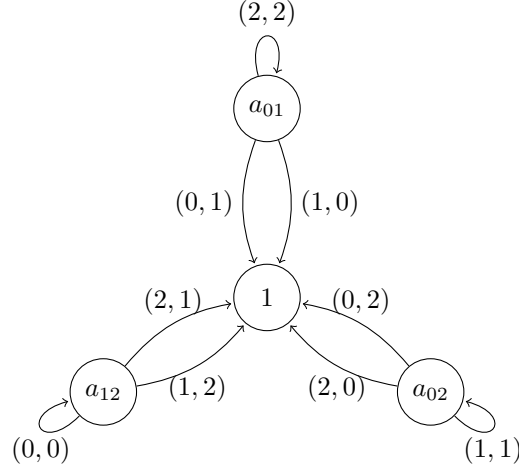


FIGURE 2.2. Nucleus of the 3-peg Hanoi Towers Group

**Proposition 2.7.** *Let  $\mathcal{J}_G$  be the limit space of a contracting action  $(G, \mathbf{X})$  with the quotient map  $p : \mathbf{X}^{-\omega} \rightarrow \mathcal{J}_G$ . Then  $p(\dots x_2 x_1) = p(\dots y_2 y_1)$  implies that  $p(\dots x_{n+1} x_n) = p(\dots y_{n+1} y_n)$  for all  $n \in \mathbb{Z}^+$ . In particular, the induced shift map  $s : \mathcal{J}_G \rightarrow \mathcal{J}_G$  defined by  $s \circ p = p \circ \sigma$  is well-defined.*

*Proof.* Since  $p(\dots x_2 x_1) = p(\dots y_2 y_1)$ , the left-infinite words  $\dots x_2 x_1$  and  $\dots y_2 y_1$  are asymptotically equivalent. Thus, there exists a left-infinite path  $\dots e_2 e_1$  in the nucleus where the label of the edge  $e_k$  is  $(x_k, y_k)$ . Then the left-infinite path  $\dots e_{n+1} e_n$  gives the asymptotic equivalence between  $\dots x_{n+1} x_n$  and  $\dots y_{n+1} y_n$ , and so  $p(\dots x_{n+1} x_n) = p(\dots y_{n+1} y_n)$ .  $\square$

Finally, we introduce the notion of a tile.

**Definition 2.8.** Let  $\mathcal{J}_G$  be the limit space of a contracting action. For each  $w \in \mathbf{X}^*$  such that  $|w| = n$ , the  $n^{\text{th}}$  level tile  $\mathcal{T}_w$  is defined as the image  $p(\mathbf{X}^{-\omega} w)$  in  $\mathcal{J}_G$ .

### 3. SELF-SIMILAR STRUCTURES ON LIMIT SPACES

An important aspect of a fractal is its self-similarity. To make the adjective “self-similar” precise, we adopt the following definition:

**Definition 3.1.** Let  $K$  be a compact metrizable topological space, and let  $F_i : K \rightarrow K$  be a continuous injection for each  $i \in \mathbf{X}$ , then the system  $\mathcal{L} = (K, \mathbf{X}, \{F_i\}_{i \in \mathbf{X}})$  is said to be a *self-similar structure on  $K$*  if there exists a continuous surjection  $\pi : \mathbf{X}^{-\omega} \rightarrow K$  such that the relation  $F_i \circ \pi = \pi \circ i$  holds, where  $i(w) = wi$  for all  $i \in \mathbf{X}$ . In this case, for each  $w = x_n \dots x_2 x_1 \in \mathbf{X}^*$ , we define  $F_w$  by  $F_w = F_{x_1} F_{x_2} \circ \dots \circ F_{x_n}$ , and the  $n^{\text{th}}$  level cell  $K_w$  to be  $K_w = F_w(K)$ .

Many well-known fractals, such as the Sierpiński gasket and the pentakun (see Figure 3.1, has self-similar structures. For more details on these structures, see Examples 6.2 and 6.5.

It is shown in [Kig01] (Proposition 1.3.3) that if  $\mathcal{L} = (K, \mathbf{X}, \{F_i\}_{i \in \mathbf{X}})$  is a self-similar structure on  $K$ , then  $\pi$  is unique. Therefore, given a self-similar structure on  $K$ , we can discuss its surjection  $\pi$ .

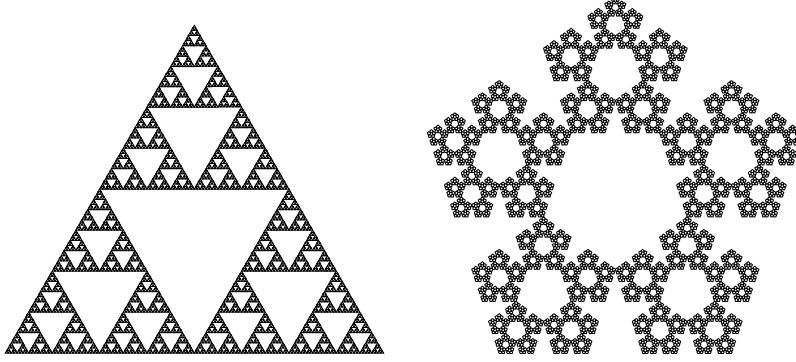


FIGURE 3.1. Sierpinski gasket and pentakun

We are interested in the self-similar structures on the limit space of a contracting action.

**Condition 3.2.** A continuous surjection  $\pi : X^{-\omega} \rightarrow K$  is said to satisfy this condition if  $\pi(\dots x_2 x_1) = \pi(\dots y_2 y_1)$  implies that  $\pi(\dots x_2 x_1 i) = \pi(\dots y_2 y_1 i)$  for each  $i \in X$ , and consequently  $\pi(\dots x_2 x_1 w) = \pi(\dots y_2 y_1 w)$  for each  $w \in X^*$ .

**Proposition 3.3.** Let  $\mathcal{J}_G$  be the limit space of a contracting action  $(G, X)$ . There exists a self-similar structure  $\mathcal{L} = (\mathcal{J}_G, X, \{F_i\}_{i \in X})$  on  $\mathcal{J}_G$ , such that the associated continuous surjection  $\pi$  is the quotient map  $p : X^{-\omega} \rightarrow \mathcal{J}_G$ , if and only if the quotient map  $p$  satisfies Condition 3.2.

*Proof.* Suppose first that  $p$  satisfies Condition 3.2; then  $p(\dots x_2 x_1) = p(\dots y_2 y_1)$  implies that  $p(\dots x_2 x_1 i) = p(\dots y_2 y_1 i)$  for each  $i \in X$ . As a consequence, if we define  $F_i$  by  $F_i \circ p = p \circ i$ , we see that  $F_i$  is well-defined and continuous. Moreover,  $F_i$  is injective for each  $i$ . Indeed, if  $F_i(p(\dots x_2 x_1)) = F_i(p(\dots y_2 y_1))$ , then  $p(\dots x_2 x_1 i) = p(\dots y_2 y_1 i)$ , and so by Proposition 2.7,  $p(\dots x_2 x_1) = p(\dots y_2 y_1)$ . Therefore,  $\mathcal{L} = (\mathcal{J}_G, X, \{F_i\}_{i \in X})$  is a self-similar structure on  $\mathcal{J}_G$  with  $p$  being the associated continuous surjection.

Conversely, suppose that there exists a self-similar structure  $\mathcal{L} = (\mathcal{J}_G, X, \{F_i\}_{i \in X})$  on  $\mathcal{J}_G$ , such that the associated continuous surjection is  $p$ . Suppose that  $p(\dots x_2 x_1) = p(\dots y_2 y_1)$ . Since  $F_w$  is well-defined for every  $w \in X^*$ , we see that

$$p(\dots x_2 x_1 w) = F_w(p(\dots x_2 x_1)) = F_w(p(\dots y_2 y_1)) = p(\dots y_2 y_1 w),$$

and so Condition 3.2 is satisfied.  $\square$

Hereafter, if we say that a limit space  $\mathcal{J}_G$  has a self-similar structure, we mean that there is a self-similar structure on  $\mathcal{J}_G$  such that the associated continuous surjection  $\pi$  is  $p$ . In particular, we shall refer to the self-similar structure defined in the proof above as *the* self-similar structure on the limit space  $\mathcal{J}_G$ .

We now wish to investigate which contracting actions have a limit space with a self-similar. We shall see that they are exactly those contracting actions satisfying the following condition.

**Condition 3.4.** A contracting action  $(G, X)$  is said to satisfy this condition if for every left-infinite path  $e = \dots e_2 e_1$  in the nucleus ending at a non-trivial state and for every  $w \in X^*$ , there exists a left-infinite path  $f = \dots f_2 f_1$  in the nucleus ending at a state  $g$ , such that the label of the edge  $f_n$  is the same as the label of  $e_n$ , and  $g(w) = w$ .

**Theorem 3.5.** *The limit space  $\mathcal{J}_G$  of a contracting action  $(G, X)$  has a self-similar structure if and only if  $(G, X)$  satisfies Condition 3.4.*

*Proof.* We are to show that  $(G, X)$  satisfies Condition 3.4 if and only if the quotient map  $p : X^{-\omega} \rightarrow \mathcal{J}_G$  satisfies Condition 3.2; then we can apply Proposition 3.3 to arrive at the desired conclusion.

Suppose first that  $(G, X)$  satisfies Condition 3.4. Let  $p(\dots x_2 x_1) = p(\dots y_2 y_1)$ , then  $\dots x_2 x_1$  is asymptotically equivalent to  $\dots y_2 y_1$ . Therefore, there exists a left-infinite path  $\dots e_2 e_1$  within the nucleus passing through the states  $\dots g_2 g_1 g_0$ , where the label of the edge  $e_n$  is  $(x_n, y_n)$ .

If  $g_0 = 1$ , then it is evident that  $\dots x_2 x_1 w$  is asymptotically equivalent to  $\dots y_2 y_1 w$  for all  $w \in X^*$ , and so  $p(\dots x_2 x_1 w) = p(\dots y_2 y_1 w)$ .

If  $g_0 \neq 1$ , then by Condition 3.4, for each  $w \in X^*$ , there exist a state  $h \in \mathcal{N}$  and a left-infinite path  $\dots f_2 f_1$  within the nucleus ending at  $h$ , such that the label of  $f_n$  is also  $(x_n, y_n)$ , and that  $h(w) = w$ . Then  $\dots x_2 x_1 w$  is asymptotically equivalent to  $\dots y_2 y_1 w$ , and so  $p(\dots x_2 x_1 w) = p(\dots y_2 y_1 w)$ . Therefore,  $p$  satisfies Condition 3.2.

Conversely, suppose that  $p$  satisfies Condition 3.2. Let  $e = \dots e_2 e_1$  be a left-infinite path in the nucleus ending at a non-trivial state, where the label of the edge  $e_n$  is  $(x_n, y_n)$ . Then  $p(\dots x_2 x_1) = p(\dots y_2 y_1)$ . By Condition 3.2,  $p(\dots x_2 x_1 w) = p(\dots y_2 y_1 w)$  for every  $w \in X^*$ . Thus, there exists a left-infinite path  $f = \dots f_2 f_1$  in the nucleus ending at the state  $g$ , such that the label of the edge  $f_n$  is also  $(x_n, y_n)$ , and  $g(w) = g$ .  $\square$

Henceforth, whenever we discuss a limit space with a self-similar structure, we shall use  $\pi$  to denote both the surjection and the quotient map.

For a limit space with a self-similar structure, the notion of a cell and a tile coincides.

**Proposition 3.6.** *Let  $\mathcal{L} = (\mathcal{J}_G, X, \{F_i\}_{i \in X})$  be the self-similar structure on the limit space  $\mathcal{J}_G$  of a contracting action  $(G, X)$  satisfying Condition 3.4. Then for each  $w \in X^*$ , the  $n^{\text{th}}$  level tile  $\mathcal{T}_w$  and the  $n^{\text{th}}$  level cell  $F_w(\mathcal{J}_G)$  are the same set.*

We end this section by proving a strengthened version of Proposition 4.4 of [NT08].

**Proposition 3.7.** *Let  $\mathcal{L} = (\mathcal{J}_G, X, \{F_i\}_{i \in X})$  be the self-similar structure on the limit space  $\mathcal{J}_G$  of a contracting action  $(G, X)$  satisfying Condition 3.4. Then the restriction of the induced shift map  $s : \mathcal{J}_G \rightarrow \mathcal{J}_G$  onto the tile  $\mathcal{T}_{wi} = F_{wi}(\mathcal{J}_G)$  is equivalent to  $F_i^{-1} : \mathcal{T}_{wi} \rightarrow \mathcal{T}_w$  for every  $w \in X^*$  and  $i \in X$ .*

*Therefore, the restriction of the induced shift map  $s$  onto the tile  $\mathcal{T}_{wi}$  is a homeomorphism  $s : \mathcal{T}_{wi} \rightarrow \mathcal{T}_w$  for every  $w \in X^*$  and  $i \in X$ . In particular, the tiles are homeomorphic to the limit space  $\mathcal{J}_G$ .*

*Proof.* It is evident that the restriction of  $s : \mathcal{J}_G \rightarrow \mathcal{J}_G$  onto the tile  $\mathcal{T}_{wi}$  is the map  $s : \mathcal{T}_{wi} \rightarrow \mathcal{T}_w$ . Consider now the restriction of  $F_i$  onto  $\mathcal{T}_w$ , which is the map  $F_i : \mathcal{T}_w \rightarrow \mathcal{T}_{wi}$ . On the set  $X^{-\omega} wi$  of left-infinite words that end in  $wi$ , we have that  $F_i \circ s \circ \pi = F_i \circ \pi \circ \sigma = \pi \circ i \circ \sigma = \pi$ , and so we see that  $s$  and  $F_i$  are inverses of each other.

The map  $F_i$  is continuous by definition. Since  $s$  is bijective and continuous, and its inverse is also continuous, we obtain that  $s : \mathcal{T}_{wi} \rightarrow \mathcal{T}_w$  is a homeomorphism.  $\square$

#### 4. THE LIMIT SPACE OF A P.C.F. ACTION

We now turn to a class of self-similar structures that is important in analysis on fractals, namely post-critically finite (or p.c.f.) structures. The criterion for the limit space of a contracting action to be finitely ramified has been shown in [BN03], and the main result in

this section (Theorem 4.7) is to apply this result to the case of limit spaces with a self-similar structure. We first follow [BN03] and adopt the following definition:

**Definition 4.1** ([BN03] Definition 5.1). A contracting action is said to be *post-critically finite*, or p.c.f. for short, if there exists only a finite number of left-infinite paths in the Moore diagram of its nucleus which end at a non-trivial state.

We then follow [Kig93, Kig01] and make the following definitions:

**Definition 4.2** ([Kig93] Definition 1.5, [Kig01] Definition 1.3.4). Let  $\mathcal{L} = (K, X, \{F_i\}_{i \in X})$  be a self-similar structure on  $K$ . The *critical set*  $\mathcal{C}$  of  $\mathcal{L}$  is defined by  $\mathcal{C}(\mathcal{L}) = \pi^{-1}(\bigcup_{i,j \in X, i \neq j} (K_i \cap K_j))$ , and the *post-critical set*  $\mathcal{P}$  is defined by  $\mathcal{P}(\mathcal{L}) = \bigcup_{n \geq 1} \sigma^n(\mathcal{C})$ , where  $\sigma$  is the shift operator on  $X^{-\omega}$ .

**Definition 4.3** ([Kig93] Definition 1.12, [Kig01] Definition 1.3.13). A self-similar structure is said to be *post-critically finite*, or p.c.f. for short, if its post critical set  $\mathcal{P}$  is finite.

To prove our main result of this section, we use a lemma from [BN03]. We need the notion of a finitely ramified set to understand the lemma.

**Definition 4.4.** The limit space of a contracting action is said to be *finitely ramified in the group-theoretical sense*, or simply *finitely ramified*, if the intersection of every two distinct tiles of the same level is finite.

A self-similar structure is said to be *finitely ramified in the fractal sense*, or simply *finitely ramified*, if the intersection of every two distinct cells of the same level is finite.

It is a standard result that a p.c.f. self-similar structure is finitely ramified. A finitely ramified limit space (in the group-theoretical sense) is what [BN03] calls a *p.c.f. limit space*. As we shall see later, it is true that the self-similar structure of a limit space is p.c.f. if and only if the limit space is finitely ramified; however, to avoid confusion with the notion of a p.c.f. fractal (in the [Kig93, Kig01] sense), we shall not use the terminology introduced by [BN03].

By Proposition 3.6, the self-similar structure  $\mathcal{L} = (\mathcal{J}_G, X, \{F_i\}_{i \in X})$  of the limit space of a contracting action is finitely ramified (in the fractal sense) if and only if  $\mathcal{J}_G$  is finitely ramified (in the group-theoretical sense).

We now quote the lemma from [BN03].

**Lemma 4.5** ([BN03] Corollary 4.2). *The limit space  $\mathcal{J}_G$  of a contracting action  $(G, X)$  is finitely ramified if and only if  $(G, X)$  is p.c.f.*

Our result justifies the use of the terminology “p.c.f.” in the “p.c.f. action” in Lemma 4.5. Before we state our main result, we first prove a useful proposition, the proof of which is used in [BN03] Corollary 4.2.

**Proposition 4.6.** *Let  $\mathcal{J}_G$  be the limit space of a contracting action  $(G, X)$ . Then for every point  $a \in \mathcal{J}_G$ , the set  $p^{-1}(a)$  is finite.*

*Proof.* We prove that each asymptotic equivalence class of a contracting action has at most  $|\mathcal{N}|$  elements, and so the quotient map  $p$  cannot map infinitely many elements to a point in  $K$ .

Given a sequence  $\dots x_2 x_1 \in X^{-\omega}$ , we denote by  $E$  the set of all left-infinite paths  $\dots e_2 e_1$  passing through the states  $\dots g_2 g_1 g_0$  within the nucleus, where the label of the edge  $e_n$  is  $(x_n, y_n)$  for some  $y_n \in X$ . We know that since  $g_{n-1} = g_n|_{x_n}$ , the state  $g_{n-1}$  in the path is uniquely determined by  $g_n$  and  $x_n$ . This shows that given two distinct paths



$e = \dots e_2 e_1$  and  $f = \dots f_2 f_1$  in  $E$ , if  $e_k = f_k$  for some  $k$ , then  $e_n = f_n$  for all  $n \leq k$ . Consequently, there exists a positive integer  $N_{ef}$  such that  $e_n \neq f_n$  for all  $n \geq N_{ef}$ .

Suppose there exist more than  $|\mathcal{N}|$  distinct left-infinite paths in  $E$ ; then we can choose a set  $F$  of  $|\mathcal{N}| + 1$  distinct paths in  $E$ . Let  $N = \max_{e,f \in F} N_{ef}$ , then for every pair of  $e, f \in F$ ,  $e_n \neq f_n$  for all  $n \geq N$ . This is a contradiction since it implies that there are more than  $|\mathcal{N}|$  states in the nucleus.  $\square$

**Theorem 4.7.** *The self-similar structure  $\mathcal{L} = (\mathcal{J}_G, \mathbf{X}, \{F_i\}_{i \in \mathbf{X}})$  on the limit space  $\mathcal{J}_G$  of a contracting action  $(G, \mathbf{X})$  is p.c.f. if and only if  $(G, \mathbf{X})$  is p.c.f.*

*Proof.* Suppose first that  $\mathcal{L}$  is p.c.f. Then  $\mathcal{L}$  is finitely ramified, and so  $\mathcal{J}_G$  is finitely ramified. Therefore, by Lemma 4.5,  $(G, \mathbf{X})$  is p.c.f.

Conversely, suppose that  $\mathcal{L}$  is not p.c.f. We can assume that  $\mathcal{L}$  is finitely ramified, for otherwise  $\mathcal{J}_G$  is not finitely ramified, and we can apply Lemma 4.5 to obtain that  $(G, \mathbf{X})$  is not p.c.f. In particular, since the image  $\pi(\mathcal{C})$  of the critical set is the intersection of cells of the first level, we see that  $\pi(\mathcal{C})$  is finite.

Moreover, if the critical set  $\mathcal{C}$  is infinite, then there exists at least one point  $a \in \pi(\mathcal{C})$  such that the set  $\pi^{-1}(a)$  is infinite. This is impossible, since by Proposition 4.6,  $\mathcal{J}_G$  cannot be a limit space. Therefore,  $\mathcal{C}$  is finite.

Now since  $\mathcal{P}$  is infinite, there exists at least one element  $x \in \mathcal{C}$  such that the shift map  $\sigma$  generates infinitely many distinct elements of  $\mathbf{X}^{-\omega}$  when repeatedly applied to  $x$ . Since  $x \in \mathcal{C}$ , there exists some  $y \in \mathcal{C}$  such that  $x_1 \neq y_1$  and  $x$  and  $y$  are asymptotically equivalent; thus there exists a left-infinite path  $\dots e_2 e_1$  passing through the states  $\dots g_2 g_1 g_0$  within the nucleus, where the label of the edge  $e_n$  is  $(x_n, y_n)$  and  $g_n$  is non-trivial for all  $n \geq 1$ .

We now show that if  $i \neq j$ , then the left-infinite paths  $\dots e_{i+1} e_i$  and  $\dots e_{j+1} e_j$ , ending at the states  $g_{i-1}$  and  $g_{j-1}$  respectively, are two distinct paths. Without loss of generality, we can assume  $i > j$ ; if the two paths are identical, then we have that  $e_{m+(i-j)} = e_m$  for all  $m \geq j$ . This would imply that  $x_{m+(i-j)} = x_m$  for all  $m \geq j$ , and consequently  $\sigma^{m+(i-j)}(x) = \sigma^m(x)$  for all  $m \geq j-1$ . But this means that the shift map  $\sigma$  only generates at most (including  $x$  itself)  $(j-1) + (i-j) = i-1$  distinct elements of  $\mathbf{X}^{-\omega}$ , contradicting the definition of  $x$ . Therefore the two paths are distinct.

Also, since for  $n \geq 1$  each  $g_n$  is a non-trivial state in the nucleus, which is finite because the action is contracting, it follows that there exists an infinite sequence  $\{n_k\}$  such that  $g_{n_k} = g$  for some non-trivial state  $g$  in the nucleus and for all  $k$ . If we now consider the left-infinite paths  $\dots e_{n_k+2} e_{n_k+1}$ , we see that these paths are pairwise distinct and all end at the state  $g$ , and so there exist infinitely many left-infinite paths in the Moore diagram of the nucleus which end at a non-trivial state.  $\square$

**Corollary 4.8.** *The self-similar structure  $\mathcal{L} = (\mathcal{J}_G, \mathbf{X}, \{F_i\}_{i \in \mathbf{X}})$  on the limit space  $\mathcal{J}_G$  of a contracting action  $(G, \mathbf{X})$  is p.c.f. if and only if it is finitely ramified. In other words, a self-similar structure that is finitely ramified but not p.c.f. cannot be a self-similar structure on the limit space of a contracting action.*

We already know a certain class of fractals that are finitely ramified but not p.c.f. that cannot arise as the limit space of any contracting action. In particular, being finitely ramified implies that the image  $\pi(\mathcal{C})$  of the critical set is finite, while not being p.c.f. implies that the post-critical set  $\mathcal{P}$  is infinite. If this is the case, Proposition 4.6 implies that fractals with either infinite  $\mathcal{C}$  or finite  $\pi(\mathcal{P})$  cannot be the result of a limit space. A simple example of such a fractal is the diamond fractal, which has been discussed in [BCD<sup>+</sup>08, HK10].

However, we now have a new class of fractals that are not limit spaces of contracting actions, namely those self-similar sets that are finitely ramified and satisfy that

- (1)  $\mathcal{C}$  is finite but  $\pi(\mathcal{C})$  is infinite; and
- (2)  $\mathcal{P}$  is infinite but  $\pi(\mathcal{P})$  is finite.

An example of such fractals is the Kameyama fractal, introduced in [Kam00] in a different setting and discussed in [Hve05].

Combining these results with those in the last section, we also have the following result.

**Corollary 4.9.** *The limit space  $\mathcal{J}_G$  of a contracting action  $(G, X)$  has a p.c.f. self-similar structure if and only if  $(G, X)$  satisfies Condition 3.4 and is p.c.f.*

Finally, we look at the related notion of a strictly p.c.f. group, which is defined and discussed in [NT08]. Its definition requires the notion of bounded automata, which is first introduced in [Sid00]. We show that the limit space of a finitely generated strictly p.c.f. group indeed has a p.c.f. self-similar structure.

**Definition 4.10.** An automorphism  $g \in \text{Aut } X^*$  is said to be *bounded* if the Moore diagram of the set  $\{g|_w : w \in X^*\}$  is finite and its oriented cycles consisting of non-trivial elements are disjoint and not connected by directed paths.

**Definition 4.11** ([NT08] Definition 4.2). A self-similar group  $(G, X)$  is said to be *strictly p.c.f.* if and only if it is a subgroup of the group  $\mathcal{B}(X)$  of bounded automorphisms and every element of the nucleus of  $G$  changes at most one letter in every word  $w \in X^*$ .

The fact that the set  $\mathcal{B}(X)$  of bounded automorphisms is indeed a group follows from the following theorem in [BN03, Nek05], which also shows the relationship between  $\mathcal{B}(X)$  and p.c.f. groups.

**Theorem 4.12** ([BN03] Theorem 5.3, [Nek05] Corollary 3.9.8). *The set  $\mathcal{B}(X)$  of all bounded automorphisms of the tree  $X^*$  is a group.*

*A finitely generated self-similar automorphism group  $G$  of the tree  $X^*$  is a p.c.f. group if and only if it is a subgroup of  $\mathcal{B}(X)$ . In particular, every finitely generated self-similar subgroup of  $\mathcal{B}(X)$  is contracting.*

**Corollary 4.13.** *The self-similar structure on the limit space  $\mathcal{J}_G$  of a finitely generated strictly p.c.f. group  $(G, X)$  is p.c.f.*

*Proof.* If  $(G, X)$  is strictly p.c.f., then it is a subgroup of  $\mathcal{B}(X)$ . By Theorem 4.12, it is p.c.f. Therefore, by Corollary 4.9, we need only to show that if  $(G, X)$  is strictly p.c.f., then it satisfies Condition 3.4. This follows from the assumption that every element of the nucleus of  $G$  changes at most one letter in every word  $w \in X^*$ . Indeed, for every  $g \in \mathcal{N}$  and  $w \in X^*$  such that  $g(w) \neq w$ , it follows that  $g|_w(v) = v$  for all  $v \in X^*$ , or in other words  $g|_w = 1$ . This implies that if  $\dots e_2 e_1$  is any left-infinite path passing through the states  $\dots g_2 g_1 g_0$  such that the label of some  $e_n$  is  $(x_n, y_n)$ , where  $x_n \neq y_n$ , then  $g_0 = 1$ . (In fact, we always have that  $n = 1$ .) Consequently, there are no left-infinite paths with non-trivial labels that do not end at the trivial state, and so Condition 3.4 is trivially satisfied.  $\square$

For an example of a p.c.f. action that satisfies Condition 3.4 but is not strictly p.c.f., see Example 6.3.

## 5. FROM A P.C.F. SELF-SIMILAR STRUCTURE

In this section, we shall be concerned with the construction of a contracting action whose limit space has a given p.c.f. self-similar structure. More precisely, we shall construct a contracting action whose limit space has a self-similar structure that is isomorphic to a given p.c.f. self-similar structure in the following sense:

**Definition 5.1** ([Kig01] Definition 1.3.2). Let  $\mathcal{L}_j = (K_j, X_j, \{F_i^{(j)}\}_{i \in X_j})$  be self-similar structures for  $j = 1, 2$ , and let  $\pi_j : X_j^{-\omega} \rightarrow K_j$  be the associated continuous surjections. The self-similar structures  $\mathcal{L}_1$  and  $\mathcal{L}_2$  are said to be *isomorphic* if there exists a bijective map  $\rho : X_1 \rightarrow X_2$  such that  $\pi_2 \circ \iota_\rho \circ \pi_1^{-1}$  is a well-defined homeomorphism between  $K_1$  and  $K_2$ , where  $\iota_\rho : X_1^{-\omega} \rightarrow X_2^{-\omega}$  is the natural bijective map defined by  $\iota(\dots x_2 x_1) = \dots \rho(x_2) \rho(x_1)$ .

Notice that given a set with a self-similar structure in general, it is possible that it is not the limit space of any contracting action. In particular, Proposition 2.7 implies that for any construction to have a hope of success, the surjection  $\pi$  of the self-similar structure must be such that if  $\pi(\dots x_2 x_1) = \pi(\dots y_2 y_1)$ , then  $\pi(\dots x_{n+1} x_n) = \pi(\dots y_{n+1} y_n)$ . An equivalent requirement is that the induced shift map  $s : \mathcal{J}_G \rightarrow \mathcal{J}_G$  defined by  $s \circ \pi = \pi \circ \sigma$  is well-defined. For example, the usual self-similar structure on the Sierpiński gasket does not satisfy this requirement, although another self-similar structure on it does; see Example 6.2 for a more detailed description.

At the same time, a property of self-similar structures will also be useful. A close inspection of the proof of Proposition 3.3 reveals that for any self-similar structure, the surjection  $\pi$  must satisfy Condition 3.2. This leads us to the following lemma.

**Lemma 5.2.** *Let  $\mathcal{L} = (K, X, \{F_i\}_{i \in X})$  be a self-similar structure on the limit space of a contracting action  $(G, X)$ . Then the associated surjection  $\pi : X^{-\omega} \rightarrow K$  satisfies the following conditions:*

- (1)  $\pi(\dots x_2 x_1) = \pi(\dots y_2 y_1)$  implies that  $\pi(\dots x_{n+1} x_n) = \pi(\dots y_{n+1} y_n)$ , for all  $n \in \mathbb{Z}^+$ ; and
- (2)  $\pi(\dots x_2 x_1) = \pi(\dots y_2 y_1)$  implies that  $\pi(\dots x_2 x_1 w) = \pi(\dots y_2 y_1 w)$ , for all  $w \in X^*$ .

In this section, we restrict ourselves to considering only self-similar structures that are isomorphic to self-similar structures on limit spaces. In particular, for the rest of this section, we shall take for granted the existence of the shift map  $s$ , and that the associated surjection  $\pi$  satisfies the conditions in Lemma 5.2.

Given a p.c.f. self-similar structure  $\mathcal{L} = (K, X, \{F_i\}_{i \in X})$ , the surjection  $\pi$  defines an equivalence relation on  $X^{-\omega}$ . We now describe a scheme to systematically write down the equivalence classes induced by  $\pi$ .

Suppose  $\pi(\dots x_2 x_1) = \pi(\dots y_2 y_1)$ , and  $x_k = y_k$  for all  $k < N$  and  $x_N \neq y_N$ . By Condition (1) above, we have that  $\pi(\dots x_{N+1} x_N) = \pi(\dots y_{N+1} y_N)$ . Then by Condition (2),  $\pi(\dots x_{N+1} x_N w) = \pi(\dots y_{N+1} y_N w)$  for all  $w \in X^*$ , which accounts for the fact that  $\pi(\dots x_2 x_1) = \pi(\dots y_2 y_1)$ . The equivalence relation can be completely characterized by equations of the form

$$\pi(\dots x_2 x_1 w) = \pi(\dots y_2 y_1 w),$$

where  $x_1 \neq y_1$ . Moreover, the equation above implies that  $\dots x_2 x_1, \dots y_2 y_1 \in \mathcal{C}$ . Since  $\mathcal{L}$  is p.c.f.,  $\mathcal{C}$  must be finite. Consequently, the equivalence relation can be characterized by finitely many such equations.

The fact that  $\mathcal{L}$  is p.c.f. also implies that elements in  $\mathcal{C}$  have a recurring tail. If we denote  $\bar{z} = \dots z z z$  where  $z = z_k \dots z_2 z_1 \in X^*$ , then the equivalence relation can be characterized by finitely many equations of the form

$$\pi(\bar{z} x_n \dots x_2 x_1 w) = \pi(\bar{z}' y_m \dots y_2 y_1 w).$$

We shall now show that  $m = n$  in the equation above. Otherwise, without loss of generality, we can let  $m > n$ . Then by Condition (1),  $\pi(\bar{z}) = \pi(\bar{z}'y_m \dots y_{n+1})$ . Then

$$\pi(\bar{z}'y_m \dots y_{n+1}) = \pi(\bar{z}) = \pi(\bar{z}z) = \pi(\bar{z}'y_m \dots y_{n+1}z),$$

by Condition (2); likewise,

$$\pi(\bar{z}'y_m \dots y_{n+1}) = \pi(\bar{z}'y_m \dots y_{n+1}z) = \pi(\bar{z}'y_m \dots y_{n+1}zz) = \dots,$$

which would imply that  $\mathcal{C}$  is infinite, again a contradiction to the fact that  $\mathcal{L}$  is p.c.f. A similar argument shows that we must have  $z = z'$ . Therefore, the equivalence relation can be characterized by finitely many equations of the form

$$\pi(\bar{z}x_n \dots x_2x_1w) = \pi(\bar{z}y_n \dots y_2y_1w).$$

Another similar argument shows that  $z_k \neq x_n$  and  $z_k \neq y_n$ . We shall assume that  $z$  is the shortest recurring word, so that it is impossible to write  $z = v^n$  for any  $n > 1$  and  $v \in X^*$ .

Next, we show that if  $\pi(\bar{z}x_n \dots x_2x_1w) = \pi(\bar{z}y_n \dots y_2y_1w)$ , then  $\pi(\bar{z}\xi_n \dots \xi_2\xi_1w) = \pi(\bar{z}x_n \dots x_2x_1w)$  whenever  $\xi_j \in \{x_j, y_j\}$  for all  $j$ . We shall show this by induction on  $n$ . The base case  $n = 1$  is trivial. Suppose this is true for  $n = m$ , then by Condition (2),

$$\begin{aligned} \pi(\bar{z}\xi_{m+1} \dots \xi_2\xi_1w) &= \pi(\bar{z}x_{m+1} \dots x_2x_1w) \\ &= \pi(\bar{z}y_{m+1} \dots y_2y_1w) \\ &= \pi(\bar{z}\xi_{m+1} \dots \xi_2y_1w), \end{aligned}$$

and therefore it is true for  $n = m + 1$ . As a consequence, we have that  $\pi(\bar{z}\xi_n \dots \xi_2\xi_1w) = \pi(\bar{z}\zeta_n \dots \zeta_2\zeta_1w)$  whenever  $\xi_j, \zeta_j \in \{x_j, y_j\}$  for all  $j$ . At this point, having classified the equivalence classes induced by the associated surjection  $\pi$  of a p.c.f. self-similar structure, we can finally write the equivalence classes in the form

$$\{\bar{z}\zeta_n \dots \zeta_2\zeta_1w \mid z, w \in X^*, \zeta_j \in S_j\}$$

for fixed  $w, z \in X^*$ , and some collection of sets  $S_j \subset X$ . We introduce the shorthand

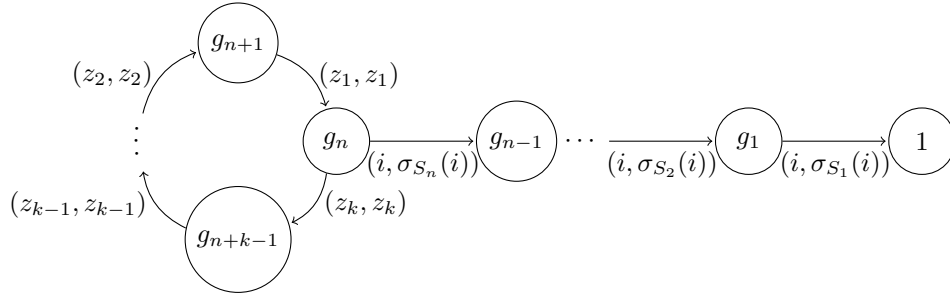
$$\bar{z}S_n \dots S_2S_1w,$$

to represent the equivalence class above.

Up to left shifts, all the equivalence classes are determined by those of the form  $\bar{z}S_n \dots S_1$ , where  $S_1$  contains more than one element. Notice that if  $\alpha$  is in the image  $\pi(\mathcal{C})$  of the critical set, then  $\pi^{-1}(\alpha) = \bar{z}S_n \dots S_1$ , for some  $z \in X^*$  and  $S_1$  with more than one element, since  $\alpha$  is in the union of the intersections of the first level cells of the space  $K$ ; conversely, the image of every equivalence class of the form  $\bar{z}S_n \dots S_1$  under  $\pi$  is a single point in  $\pi(\mathcal{C})$ . Therefore, the equivalence classes can be labeled by the finitely many elements of  $\pi(\mathcal{C})$ .

Every equivalence class of the form  $\bar{z}S_n \dots S_2S_1$  has to satisfy three properties. First, by what we discussed above, we see that if  $z = z_k \dots z_2z_1$ , then  $z_k \notin S_n$ . Second, if  $\bar{z}S_n \dots S_2S_1$  is in the list, then by Condition (1), we must have that  $\bar{z}S_n \dots S_{m+1}S_m$  is also in the list for all  $m \leq n$ . The third property is the proposition below:

**Proposition 5.3.** *Let  $\alpha$  and  $\beta$  be distinct elements of  $\pi(\mathcal{C})$  of a p.c.f. self-similar structure, such that  $\pi^{-1}(\alpha)$  and  $\pi^{-1}(\beta)$  have the same recurring tail  $\bar{z}$ . Let  $\pi^{-1}(\alpha) = \bar{z}S_n \dots S_2S_1$  and  $\pi^{-1}(\beta) = \bar{z}T_m \dots T_2T_1$ . If  $S_{n-k} = T_{m-k}$  for all  $0 \leq k < N$ , then either  $S_{n-N} = T_{m-N}$  or  $S_{n-N} \cap T_{m-N} = \emptyset$ .*

FIGURE 5.1. The generators corresponding to  $\alpha = \pi(\bar{z}S_n \dots S_2S_1)$ 

*Proof.* Suppose  $S_{n-N} \cap T_{m-N} \neq \emptyset$ , and let  $x \in S_{n-N} \cap T_{m-N}$ . Then for all  $s_{n-N} \in S_{n-N}$  and  $t_{m-N} \in T_{m-N}$ , we have

$$\begin{aligned} \pi(\bar{z}\xi_n \dots \xi_{n-N+1}s_{n-N}\xi_{n-N-1} \dots \xi_2\xi_1) &= \pi(\bar{z}\xi_n \dots \xi_{n-N+1}x\xi_{n-N-1} \dots \xi_2\xi_1) \\ &= \pi(\bar{z}\xi_n \dots \xi_{n-N+1}t_{m-N}\xi_{n-N-1} \dots \xi_2\xi_1), \end{aligned}$$

whenever  $\xi_j \in S_j$ . Therefore, we see that  $T_{m-N} \subset S_{n-M}$ . Similarly, we obtain that  $S_{n-N} \subset T_{m-N}$ , and so  $S_{n-N} = T_{m-N}$ .  $\square$

We can now construct the desired contracting group. For each  $\alpha \in \pi(\mathcal{C})$ , we can write  $\pi^{-1}(\alpha) = \bar{z}S_n \dots S_2S_1$ , where  $z = z_k \dots z_2z_1$ . If  $S_j = \{s_1^{(j)}, s_2^{(j)}, \dots, s_m^{(j)}\}$  with  $s_i^{(j)} < s_{i+1}^{(j)}$ , we define  $\sigma_{S_j}$  to be the permutation  $(s_1^{(j)} s_2^{(j)} \dots s_m^{(j)})$ . We define  $n+k-1$  group elements as follows:

We define  $g_{\alpha,1}$  by the wreath recursion  $\sigma_{S_1}(1, \dots, 1)$ . For  $2 \leq j \leq n-1$ , we define  $g_{\alpha,j}$  to be the element whose action on  $x \in X$  is given by

$$g_{\alpha,j} \cdot x = \begin{cases} \sigma_{S_j}(x) \cdot g_{\alpha,j-1} & \text{if } x \in S_j, \\ x \cdot 1 & \text{if } x \notin S_j, \end{cases}$$

so that we have the wreath recursion  $g_{\alpha,j} = \sigma_{S_j}(1, \dots, g_{\alpha,j-1}, \dots, g_{\alpha,j-1}, \dots, 1)$ .

For  $j = n$ , we define  $g_{\alpha,n}$  by

$$g_{\alpha,n} \cdot x = \begin{cases} \sigma_{S_n}(x) \cdot g_{\alpha,n-1} & \text{if } x \in S_n, \\ x \cdot g_{\alpha,n+k-1} & \text{if } x = z_k, \\ x \cdot 1 & \text{otherwise,} \end{cases}$$

so that we have the wreath recursion  $g_{\alpha,n} = \sigma_{S_n}(1, \dots, g_{\alpha,n-1}, \dots, g_{\alpha,n-1}, \dots, g_{\alpha,n+k-1}, \dots, 1)$ .

Finally, for  $n+1 \leq j \leq n+k-1$ , we define  $g_{\alpha,j}$  by

$$g_{\alpha,j} \cdot x = \begin{cases} x \cdot g_{\alpha,j-1} & \text{if } x = z_{j-n}, \\ x \cdot 1 & \text{if } x \neq z_{j-n}, \end{cases}$$

so that we have the wreath recursion  $g_{\alpha,j} = (1, \dots, g_{\alpha,j-1}, \dots, 1)$ .

We call  $g_{\alpha,1}, \dots, g_{\alpha,n-1}$  *Type I generators* and  $g_{\alpha,n}, \dots, g_{\alpha,n+k-1}$  *Type II generators*.

The Moore diagram of these generators, which correspond to a single  $\alpha \in \pi(\mathcal{C})$ , is shown in Figure 5.1, in which the subscript  $\alpha$  has been suppressed. The label  $(i, \sigma_{S_j}(i))$  for the arrows out of  $g_n$  and the Type II generators applies to all  $i \in S_j$  and only those letters. For each generator, we have suppressed the arrows into the identity element, whose labels are  $(i, i)$  for each  $i$  that has not been shown in the diagram.

The desired group  $G_{\mathcal{L}}$  is the group generated by all the elements defined above for all  $\alpha \in \pi(\mathcal{C})$ .

**Proposition 5.4.** *Let  $\mathcal{L} = (K, X, \{F_i\}_{i \in X})$  be a p.c.f. self-similar structure. The group  $G_{\mathcal{L}}$  that is constructed by the method above is a subgroup of  $\mathcal{B}(X)$ . In particular,  $G_{\mathcal{L}}$  is contracting and p.c.f.*

*Proof.* It can easily be seen from the Moore diagram above that all the generators of  $G_{\mathcal{L}}$  are bounded automorphisms. Since  $G_{\mathcal{L}}$  is finitely generated, it follows from Theorem 4.12 that  $G_{\mathcal{L}}$  is p.c.f.  $\square$

For concrete examples illustrating our construction above, see Section 6, and in particular Examples 6.4, 6.5 and 6.6.

We claim that the self-similar structure  $\mathcal{L}' = (\mathcal{J}_{G_{\mathcal{L}}}, X, \{F'_i\}_{i \in X})$  on the limit space of  $G_{\mathcal{L}}$  is isomorphic to  $\mathcal{L}$  (Theorem 5.8). We begin the proof with a lemma, where we show that the associated continuous surjection of a self-similar structure is in fact a quotient map.

**Lemma 5.5.** *Let  $\mathcal{L} = (K, X, \{F_i\}_{i \in X})$  be a self-similar structure. The associated continuous surjection  $\pi : X^{-\omega} \rightarrow K$  of  $\mathcal{L}$  is a quotient map.*

*Proof.* We shall show that  $\pi$  is a closed map. Since  $X^{-\omega}$  is compact, every closed subset  $C$  is compact. Then  $\pi(C)$  is a compact subset of  $K$ , which is metrizable and thus Hausdorff, implying that  $\pi(C)$  is closed.  $\square$

**Lemma 5.6.** *Let  $G_{\mathcal{L}}$  be the contracting group constructed by the method above. If  $h \in G_{\mathcal{L}}$  can be written as a product  $g_m \dots g_2 g_1$  of generators of minimal length, and if  $h|_v = h$  for some non-empty word  $v = v_n \dots v_2 v_1 \in X^*$ , then  $g_j(v) = v$  and  $g_j|_v = g_j$  for all  $j$ . In particular,  $g_j$  is of Type II for all  $j$ , and  $h(v) = v$ .*

*Proof.* We denote by  $d(g)$  the minimal length of generators needed to represent an element  $g \in G_{\mathcal{L}}$ . In particular,  $d(h) = m$ . Notice that, since  $h|_x = g_m|_{g_{m-1} \dots g_1(x)} \dots g_2|_{g_1(x)} g_1|_x$ , and  $g_j|_y$  is 1 or another generating element for all  $y \in X$ , that  $d(h|_x) \leq d(h)$ . We see  $d(h) = d(h|_v) \leq d(h|_{v_n \dots v_{n-i}}) \leq d(h)$  for all  $0 < i \leq n$ , so  $d(h|_{v_n \dots v_{n-i}}) = d(h)$  for all such  $i$ .

We show that  $g_j$  is of Type II for all  $j$ . If  $g_j$  is of Type I for some  $j \leq m$ , then there exists  $N$  such that  $g_j|_w = 1$  whenever  $|w| = N$ . Since if  $g$  is a generator, then  $g|_x$  is either a generator or the identity for all  $x \in X$ , it follows that  $d(h|_w) < m$  whenever  $|w| = N$ . This is a contradiction since  $h|_v = h$  implies that for each  $k$ , there exists some  $w \in X^k$  such that  $d(h|_w) = m$ .

By the same argument, each representation of

$$h|_{v_n} = (g_m \dots g_1)|_{v_n} = g_m|_{g_{m-1} \dots g_1(v_n)} \dots g_1|_{v_n}$$

of length  $m$  consists only of Type II generators, which implies that  $g_1|_{v_n}$  is also of Type II. This is only possible if  $g_1(v_n) = v_n$ , since for any generator  $g$  and any  $x \in X$ ,  $g(x) \neq x$  implies that  $g|_x$  is either Type I or the identity. Similarly,  $g_j(v_n) = v_n$  and  $g_j|_{v_n}$  is of Type II for all  $j$ , and

$$h|_{v_n} = g_m|_{g_{m-1} \dots g_1(v_n)} \dots g_1|_{v_n} = g_m|_{v_n} \dots g_1|_{v_n}.$$

Inductively,  $g_j(v) = v$  and  $g_j|_v$  is of Type II for each  $j$ . Moreover,  $g_j|_v$  and  $g_j$  are generators corresponding to the same  $\alpha \in \pi(\mathcal{C})$ . Since there are only finitely many Type II generators corresponding to  $\alpha$ , it follows that  $g_j|_{v^l} = g_j$  for some minimal  $l$ .

Suppose  $l > 1$ . If  $g_j$  corresponds to  $\alpha \in \pi(\mathcal{C})$ , then  $\pi^{-1}(\alpha) = \bar{z} S_n \dots S_2 S_1$ , where  $z = v^l$ . This is a contradiction since we assume  $z$  to be the shortest recurring word.  $\square$

**Lemma 5.7.** *Let  $\mathcal{L} = (K, X, \{F_i\}_{i \in X})$  be a p.c.f. self-similar structure, with  $\pi : X^{-\omega} \rightarrow K$  as the associated surjection. Let  $G_{\mathcal{L}}$  be the contracting group constructed by the method above, and let  $\mathcal{J}_{G_{\mathcal{L}}}$  be its limit space. Then the quotient map  $p : X^{-\omega} \rightarrow \mathcal{J}_{G_{\mathcal{L}}}$  is such that*

$$p(\dots x_2 x_1) = p(\dots y_2 y_1)$$

*if and only if*

$$\pi(\dots x_2 x_1) = \pi(\dots y_2 y_1).$$

*Proof.* Suppose first that  $\pi(\dots x_2 x_1) = \pi(\dots y_2 y_1)$ , where  $\dots x_2 x_1 \neq \dots y_2 y_1$ . Without loss of generality, we can instead write  $\pi(\bar{z}x_n \dots x_2 x_1 w) = \pi(\bar{z}y_n \dots y_2 y_1 w)$ , with  $z = z_k \dots z_2 z_1$  and  $x_1 \neq y_1$ . All but finitely many truncations of the left-infinite word  $\bar{z}x_n \dots x_2 x_1 w$  are of the form  $z_l \dots z_2 z_1 z^N x_n \dots x_2 x_1 w$ , where  $1 \leq l \leq k$  and  $N \geq 0$ . We claim that for each word in this form, there exists an element  $g$  such that  $g(z_l \dots z_2 z_1 z^N x_n \dots x_2 x_1 w) = z_l \dots z_2 z_1 z^N y_n \dots y_2 y_1 w$ . We shall proceed by induction on  $n$ .

Consider the case when  $n = 1$ . There exists  $\alpha \in \pi(\mathcal{C})$  with  $\pi^{-1}(\alpha) = \bar{z}S_1$ , such that  $x_1, y_1 \in S_1$ . By construction, the action of  $g_{\alpha, l+1}$  on  $z_l \dots z_2 z_1 z^N x_1 w$  is

$$\begin{aligned} g_{\alpha, l+1}(z_l \dots z_2 z_1 z^N x_1 w) &= z_l g_{\alpha, l}(z_{l-1} \dots z_2 z_1 z^N x_1 w) \\ &= z_l z_{l-1} g_{\alpha, l-1}(z_{l-2} \dots z_2 z_1 z^N x_1 w) \\ &\vdots \\ &= z_l \dots z_2 z_1 z^N g_{\alpha, 1}(x_1 w) \\ &= z_l \dots z_2 z_1 z^N \sigma_{S_1}(x_1) w. \end{aligned}$$

There exists  $a \in \mathbb{N}$  such that  $\sigma_{S_1}^a(x_1) = y_1$ . Then

$$g_{\alpha, l+1}^a(z_l \dots z_2 z_1 z^N x_1 w) = z_l \dots z_2 z_1 z^N \sigma_{S_1}^a(x_1) w = z_l \dots z_2 z_1 z^N y_1 w.$$

Suppose now that the statement holds for  $n = m$ , and consider the case when  $n = m + 1$ . There exists  $\beta \in \pi(\mathcal{C})$  such that  $\pi^{-1}(\beta) = \bar{z}S_{m+1} \dots S_2 S_1$ , such that  $x_k, y_k \in S_k$  for each  $k$ . By construction, the action of  $g_{\beta, l+m+1}$  on  $z_l \dots z_2 z_1 z^N x_{m+1} \dots x_2 x_1 w$  is

$$\begin{aligned} g_{\beta, l+m+1}(z_l \dots z_2 z_1 z^N x_{m+1} \dots x_2 x_1 w) &= z_l g_{\beta, l+m}(z_{l-1} \dots z_2 z_1 z^N x_{m+1} \dots x_2 x_1 w) \\ &\vdots \\ &= z_l \dots z_2 z_1 z^N g_{\beta, m+1}(x_{m+1} \dots x_2 x_1 w) \\ &= z_l \dots z_2 z_1 z^N \sigma_{S_{m+1}}(x_{m+1}) \dots \sigma_{S_1}(x_1) w \end{aligned}$$

There exists  $b \in \mathbb{N}$  such that  $\sigma_{S_1}^b(x_1) = y_1$ . Then

$$g_{\beta, l+m+1}^b(z_l \dots z_1 z^N x_{m+1} \dots x_1 w) = z_l \dots z_1 z^N \sigma_{m+1}^b(x_{m+1}) \dots \sigma_2^b(x_2) y_1 w.$$

By the induction hypothesis, there exists an element  $h$  that maps this to

$$z_l \dots z_1 z^N y_{m+1} \dots y_2 y_1 w;$$

then  $h g_{\beta, l+m+1}^b$  is the desired element.

Notice that in both cases above, the element fulfilling our claim is independent of  $N$ ; therefore, the set of elements fulfilling our claim for each truncation of the left-infinite word  $\bar{z}x_n \dots x_2 x_1 w$  is finite, and thus  $p(\bar{z}x_n \dots x_1 w) = p(\bar{z}y_n \dots y_1 w)$ .

Conversely, suppose  $p(\dots x_2 x_1) = p(\dots y_2 y_1)$ , where  $\dots x_2 x_1 \neq \dots y_2 y_1$ . Then there exists a left-infinite path  $\dots e_2 e_1$  in  $\mathcal{N}$  passing through the states  $\dots h^{(2)} h^{(1)} h^{(0)}$ , such that the label of  $e_k$  is  $(x_k, y_k)$ . We can represent each  $h^{(k)}$  as a product  $g_{m_k}^{(k)} \dots g_1^{(k)}$  of

generators with minimal length  $m_k$ . If  $g$  is a generator, then  $g|_x$  is either another generator or the identity for all  $x \in X$ ; therefore,  $m_k \leq m_{k+1}$  for all  $k$ . At the same time, since the nucleus  $\mathcal{N}$  is finite,  $m_k$  is uniformly bounded from above. Therefore, there exists some minimal  $M$  such that  $m_k = m$  for some constant  $m$  for all  $k \geq M$ .

By Proposition 5.4,  $G_{\mathcal{L}}$  is p.c.f. Therefore, the path  $\dots e_2 e_1$  must have a recurring tail; that is, there exists some minimal  $N \geq M$  and some minimal  $r \geq 1$  such that  $e_{k+r} = e_k$  for all  $k > N$ . Then  $h^{(k+r)} = h^{(k)}$  for all  $k \geq N$ . If we write  $z = z_r \dots z_1 = x_{N+r} \dots x_{N+1}$ , then we have

$$p(\bar{z}x_N \dots x_2 x_1) = p(\overline{h^{(N)}(z)} y_N \dots y_2 y_1).$$

Now we also have that  $h^{(N)}|_z = h^{(N)}$ . By Lemma 5.6, it follows that  $g_j^{(N)}(z) = z$  and  $g_j^{(N)}|_z = g_j^{(N)}$  for all  $j \leq m$ , and  $h^{(N)}(z) = z$ .

Then our original equation becomes

$$p(\bar{z}x_N \dots x_2 x_1) = p(\bar{z}y_N \dots y_2 y_1),$$

where  $y_n \dots y_2 y_1 = g_m^{(N)} \dots g_1^{(N)}(x_n \dots x_2 x_1)$ .

Consider the generator  $g_1^{(N)}$ . Since  $g_1^{(N)}(z) = z$  and  $g_1^{(N)}|_z = g_1^{(N)}$ , there exists a left-infinite path  $\dots f_2 f_1$  ending at  $g_1^{(N)}$ , where the label of  $f_{r+s}$  is  $(z_s, z_s)$ , for  $s < r$ . Therefore,

$$p(\bar{z}x_N \dots x_2 x_1) = p(\bar{z}g_1^{(N)}(x_N \dots x_2 x_1)).$$

Similarly, we obtain,

$$p(\bar{z}x_N \dots x_2 x_1) = p(\bar{z}g_1^{(N)}(x_N \dots x_2 x_1)) = \dots = p(\bar{z}g_m^{(N)} \dots g_1^{(N)}(x_N \dots x_2 x_1)),$$

from which we deduce that  $g_j^{(N)} \dots g_1^{(N)}(x_N) \neq z_r$  for all  $j$ .

We now show that the above equation will continue to hold if we replace  $p$  by  $\pi$ . For each  $j$ , we can write  $g_j^{(N)} = g_{\alpha_j, l_j}$  for some  $\alpha_j \in \pi(\mathcal{C})$  with  $\pi^{-1}(\alpha_j) = z^{(j)} S_{j, n_j} \dots S_{j, 2} S_{j, 1}$ , where  $l_j \geq n_j$ , since  $g_j^{(N)}$  is of Type II by Lemma 5.6. If  $l_j \neq n_j$ , then  $g_j^{(N)}|_x = 1$  for all  $x \neq z_r$ . Therefore,

$$g_j^{(N)}(g_{j-1}^{(N)} \dots g_1^{(N)}(x_N)) = g_{j-1}^{(N)} \dots g_1^{(N)}(x_N)$$

and

$$g_j^{(N)}|_{(g_{j-1}^{(N)} \dots g_1^{(N)}(x_N))} = 1.$$

Then

$$g_j^{(N)}(g_{j-1}^{(N)} \dots g_1^{(N)}(x_N \dots x_1)) = g_{j-1}^{(N)} \dots g_1^{(N)}(x_N \dots x_1).$$

It is then trivially true that

$$\pi(\bar{z}g_j^{(N)}(g_{j-1}^{(N)} \dots g_1^{(N)}(x_N \dots x_1))) = \pi(\bar{z}g_{j-1}^{(N)} \dots g_1^{(N)}(x_N \dots x_1)).$$

Suppose now that  $l_j = n_j$ ; then by construction,  $z^{(j)}$  is the unique word of minimal length such that  $g_j^{(N)}|_{z^{(j)}} = g_j^{(N)}$ . Therefore, we see that  $z^{(j)} = z$ . If we let  $c$  be the smallest number such that  $g_j^{(N)}|_{g_{j-1}^{(N)} \dots g_1^{(N)}(x_N \dots x_c)} \neq 1$ , then by construction,

$$(g_{j-1}^{(N)} \dots g_1^{(N)})|_{x_N \dots x_{k+1}}(x_k) \in S_{j, n_j - N + k}, \text{ and} \\ g_j^{(N)}|_{g_{j-1}^{(N)} \dots g_1^{(N)}(x_N \dots x_{k+1})}((g_{j-1}^{(N)} \dots g_1^{(N)})|_{x_N \dots x_{k+1}}(x_k)) \in S_{j, n_j - N + k},$$



whenever  $c \leq k \leq N$ . But  $\bar{z}S_{j,n_j} \dots S_{j,2}S_{j,1}w$  is an equivalence class induced by  $\pi$ , and this implies that  $\bar{z}S_{j,n_j} \dots S_{j,n_j-N+c}w$  is an equivalence class, and so again we have

$$\pi(\bar{z}g_j^{(N)}(g_{j-1}^{(N)} \dots g_1^{(N)}(x_N \dots x_1))) = \pi(\bar{z}g_{j-1}^{(N)} \dots g_1^{(N)}(x_N \dots x_1)).$$

Therefore, inductively, we have shown that

$$\pi(\bar{z}x_N \dots x_2x_1) = \pi(\bar{z}y_N \dots y_2y_1),$$

which is what we wanted to prove.  $\square$

**Theorem 5.8.** *Let  $\mathcal{L} = (K, X, \{F_i\}_{i \in X})$  be a p.c.f. self-similar structure, and let  $G_{\mathcal{L}}$  be the contracting group constructed by the method above. Then the limit space  $\mathcal{J}_{G_{\mathcal{L}}}$  of  $G_{\mathcal{L}}$  has a self-similar structure  $\mathcal{L}' = (\mathcal{J}_{G_{\mathcal{L}}}, X, \{F'_i\}_{i \in X})$ . Moreover,  $\mathcal{L}'$  is isomorphic to  $\mathcal{L}$ .*

*Proof.* We know that  $\pi$  satisfies Condition 3.2. By Lemma 5.7, we see that  $p$  also satisfies Condition 3.2. Therefore, by Proposition 3.3,  $\mathcal{J}_{G_{\mathcal{L}}}$  has a self-similar structure  $\mathcal{L}'$ .

To show that  $\mathcal{L}'$  is isomorphic to  $\mathcal{L}$ , consider the map  $p \circ \pi^{-1}$ . Given an open set  $U$  in  $\mathcal{J}_{G_{\mathcal{L}}}$ ,  $p^{-1}(U)$  is open because  $p$  is continuous. Since by Lemma 5.5  $\pi$  is a quotient map, and by Lemma 5.7  $p^{-1}(U)$  is saturated, it follows that  $\pi \circ p^{-1}(U)$  is open. Therefore,  $p \circ \pi^{-1}$  is continuous. Since  $p$  is by definition a quotient map, we obtain that the inverse is also continuous. Thus,  $p \circ \pi^{-1}$  is a well-defined homeomorphism between  $\mathcal{J}_{G_{\mathcal{L}}}$  and  $K$ .  $\square$

In the construction above, we have only assumed that the induced shift map  $s$  with regard to the self-similar structure is well-defined, which is a necessary condition for any construction to succeed. By Theorem 5.8, it is also the sufficient condition; therefore, we have the following theorem.

**Theorem 5.9.** *Let  $\mathcal{L} = (K, X, \{F_i\}_{i \in X})$  be a p.c.f. self-similar structure, with an associated continuous surjection  $\pi : X^{-\omega} \rightarrow K$ . Then there exists a contracting action  $(G, X)$  such that its limit space  $\mathcal{J}_G$  has a self-similar structure  $\mathcal{L}' = (\mathcal{J}_G, X, \{F'_i\}_{i \in X})$  that is isomorphic to  $\mathcal{L}$ , if and only if the induced shift map  $s : K \rightarrow K$  defined by  $s \circ \pi = \pi \circ \sigma$  is well-defined.*

## 6. EXAMPLES

**Example 6.1** (The binary adding machine). One of the most basic examples of a self-similar action is the binary adding machine. It is the group  $G$  generated by the element  $a = (01)(1, a)$ , acting on the binary tree (i.e.  $X = \{0, 1\}$ ). The action of  $a$  can be thought of as adding 1 to the last digit of the (left-handed) binary representation of a real number.

The nucleus of  $G$  is  $\{1, a, a^{-1}\}$ , which is depicted in Figure 2.1. The asymptotic equivalence is given by  $\bar{0}1w \sim \bar{1}0w$  for all  $w \in X^*$  and  $\bar{0} \sim \bar{1}$ . Therefore, the limit space  $\mathcal{J}_G$  is homeomorphic to the circle  $\mathbb{R}/\mathbb{Z}$ , where each point on the circle corresponds to its (left-infinite) binary expansion.

The action  $(G, X)$  does not satisfy Condition 3.4, and therefore its limit space does not have the naturally induced self-similar structure. To see this, notice that  $\bar{0} \sim \bar{1}$  but  $\bar{0} \not\sim \bar{1}0$ , and so  $F : v \mapsto v0$  is not a well defined map on  $X^{-\omega}$ , which shows the non-existence of a self-similar structure on  $\mathbb{R}/\mathbb{Z}$ . Therefore, although  $G$  is clearly p.c.f., its limit space does not have a p.c.f. self-similar structure.

**Example 6.2** (Sierpiński gasket). The Sierpiński gasket is typically defined, e.g. in [Kig01], as the unique non-empty compact space  $K \subset \mathbb{C}$  that is invariant under the injections  $f_j(z) = (z - p_j)/2 + p_j$ , where  $p_j$  are the vertices of an equilateral triangle. Writing

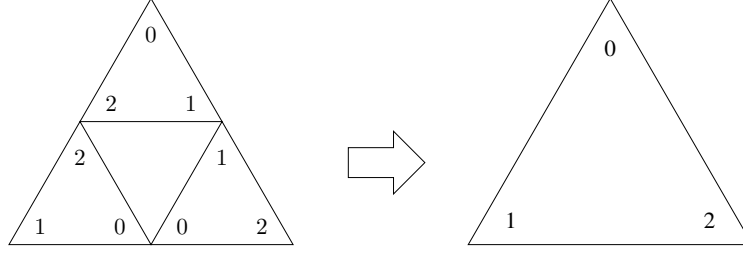


FIGURE 6.1. Self-covering of Sierpiński gasket in Example 6.2

$X = \{0, 1, 2\}$ , it has the natural p.c.f. self-similar structure  $\mathcal{L}_0 = (K, X, \{f_j\}_{j=0}^2)$ . The associated surjection  $\pi : X^{-\omega} \rightarrow K$  induces the equivalence relations

$$\bar{0}1w \sim \bar{1}0w, \quad \bar{1}2w \sim \bar{2}1w, \quad \text{and} \quad \bar{2}0w \sim \bar{0}2w$$

for all  $w \in X^*$ . However, since  $\bar{2}0 \sim \bar{0}2$  but  $\bar{2} \not\sim \bar{0}$ , the shift map  $s$  is not well defined, and so  $\mathcal{L}_0$  is not a self-similar structure on the limit space of a contracting group.

On the other hand, there is an alternate p.c.f. self-similar structure on the Sierpiński gasket, given by  $\mathcal{L} = (K, X, \{F_j\}_{j=0}^2)$ , where  $F_j = r_j \circ f_j$ , and  $r_j$  is the reflection about the axis of symmetry through  $p_j$ . This can be described by the self-covering depicted in Figure 6.1. With this self-similar structure, the induced equivalence relations are

$$\bar{0}1w \sim \bar{0}2w, \quad \bar{1}2w \sim \bar{1}0w, \quad \text{and} \quad \bar{2}0w \sim \bar{2}1w,$$

which are the relations describing the asymptotic equivalence of the 3-peg Hanoi Towers Group  $G$  [GŠ06, GŠ08]. In particular,  $\mathcal{L}$  is the natural self-similar structure on  $\mathcal{J}_G$ , where  $G$  is the group generated by the elements

$$\begin{aligned} a_{01} &= (01)(1, 1, a_{01}), \\ a_{12} &= (12)(a_{12}, 1, 1), \\ a_{20} &= (02)(1, a_{02}, 1), \end{aligned}$$

acting on the rooted tree  $X^*$ . The Moore diagram of the nucleus of the group is given in Figure 2.2. It is interesting to note that this is exactly the group  $G_{\mathcal{L}}$  that would result from our construction described in Section 5.

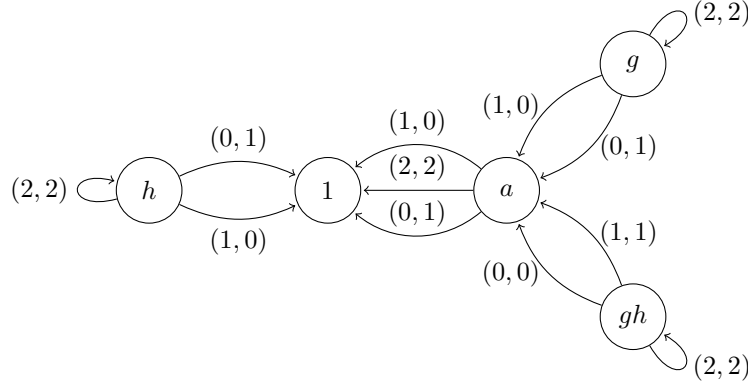
**Example 6.3.** This example highlights the fact that the p.c.f. condition and Condition 3.4 together still do not imply the strictly p.c.f. condition, by describing a group that satisfies the former conditions but not the latter.

We consider the group  $G$  whose nucleus is illustrated in Figure 6.2. This group is contracting, p.c.f., and satisfies Condition 3.4, but is not strictly p.c.f.

$G$  is generated by

$$\begin{aligned} a &= (01)(1, 1, 1), \\ g &= (01)(a, a, g), \\ h &= (01)(1, 1, h), \end{aligned}$$

and acts on  $X = \{0, 1, 2\}$ . Since all of its generators are bounded,  $G$  is bounded and thus contracting and p.c.f., with the nucleus being  $\mathcal{N} = \{1, a, g, h, gh\}$ . From the diagram, we see that  $G$  satisfies Condition 3.4, and therefore there exists a self-similar structure on its

FIGURE 6.2. Nucleus of the group  $G$  in Example 6.3

limit space  $\mathcal{J}_G$ . However, since  $g$  can change more than one letter in a give word, this group is not strictly p.c.f.

**Example 6.4.** This is a straightforward example that illustrates our construction of a contracting group with a given self-similar structure.

Consider the unit interval  $I = [0, 1]$ . Set  $F_0(x) = -(1/2)x + 1/2$ , and  $F_1(x) = (1/2)x + 1/2$ . Then  $\mathcal{L} = (I, \{0, 1\}, \{F_i\}_{i \in \{0, 1\}})$  is a p.c.f. self-similar structure on  $I$ . Notice that this self-similar structure is naturally isomorphic to a self-similar structure on the Koch curve. The critical set is given by  $\mathcal{C} = \{\bar{1}00, \bar{1}01\}$ , and its image in  $I$  is  $\pi(\mathcal{C}) = \{1/2\}$ .

There exists a well-defined continuous induced shift map  $s : I \rightarrow I$ , defined by  $s \circ \pi = \pi \circ \sigma$ ; we can write it explicitly as

$$s(x) = \begin{cases} 1 - 2x & \text{if } x \in [0, \frac{1}{2}], \\ 2x - 1 & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

By Theorem 5.9, the existence of  $s$  guarantees the success of the construction of a contracting group  $G_{\mathcal{L}}$ . There is only one entry in the list of equivalence classes, namely

$$\pi^{-1}\left(\frac{1}{2}\right) = \bar{1}S_2S_1,$$

where  $S_2 = \{0\}$  and  $S_1 = \{0, 1\}$ . We define

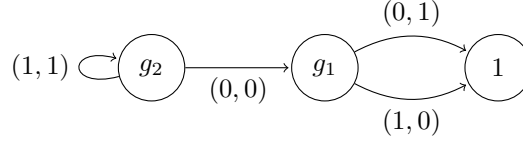
$$\begin{aligned} g_1 &= (01)(1, 1), \\ g_2 &= (g_1, g_2), \end{aligned}$$

where we have suppressed the subscript  $1/2$ . Define  $G_{\mathcal{L}} = \langle g_1, g_2 \rangle$ . Notice that

$$\begin{aligned} g_1^2 &= (1, 1) = 1, \\ g_2^2 &= (g_1^2, g_2^2) = (1, g_2^2) = 1, \end{aligned}$$

so both generators are of order 2. Moreover,

$$\begin{aligned} g_1g_2 &= (01)(g_1, g_2), \\ g_2g_1 &= (01)(g_2, g_1), \end{aligned}$$

FIGURE 6.3. Nucleus of the group  $G_{\mathcal{L}}$  in Example 6.4.

so  $G_{\mathcal{L}}$  is the infinite dihedral group, and we see that the nucleus  $\mathcal{N} = \{1, g_1, g_2\}$ . Figure 6.3 shows the Moore diagram of the nucleus. It can easily be seen that  $G_{\mathcal{L}}$  is in fact strictly p.c.f.

The limit space of the Grigorchuk group (introduced and discussed in [Gri80, Gri84]) has the same self-similar structure as  $\mathcal{L}$ . The Grigorchuk group is also strictly p.c.f. The nucleus of the Grigorchuk group is different from the nucleus of  $G_{\mathcal{L}}$ , and so they are not isomorphic. In other words, it is possible for two p.c.f. groups satisfying Condition 3.4 that are not isomorphic to each other to have limit spaces with isomorphic self-similar structures.

It has been shown that there are countably many groups whose limit space admits a self-similar structure isomorphic to  $\mathcal{L}$ ; for a classification of all such groups, see [Nek03, Šun07].

**Example 6.5** (Pentakun). The pentakun, as described in [Kig01], is the unique non-empty compact space  $K \subset \mathbb{C}$  that is invariant under the injections

$$f_k(z) = \frac{3 - \sqrt{5}}{2}(z - p_k) + p_k, \quad \text{where } p_k = e^{2\pi i k/5}.$$

Figure 3.1 gives the (rotated) picture of the pentakun.

Identifying  $X = \{0, 1, 2, 3, 4\}$ , the natural p.c.f. self-similar structure is given by  $\mathcal{L}_0 = (K, X, \{f_j\}_{j=0}^4)$ . The equivalence classes induced by  $\pi$  are

$$\bar{2}0w \sim \bar{4}1w, \quad \bar{3}1w \sim \bar{0}2w, \quad \bar{4}2w \sim \bar{1}3w, \quad \bar{0}3w \sim \bar{2}4w, \quad \text{and} \quad \bar{1}4w \sim \bar{3}0w$$

for all  $w \in X^*$ .

As with the Sierpiński gasket, the shift map is not defined for this self-similar structure, and  $\mathcal{L}_0$  is not the self-similar structure on a limit space. However, like the Sierpiński gasket, there is a modified p.c.f. self-similar structure that can be achieved as the self-similar structure on the limit space of a contracting group.

Consider  $\mathcal{L} = (K, X, \{F_j\}_{j=0}^4)$ , where  $F_j = r_j \circ f_j$ , and  $r_j$  is the reflection about the line joining  $p_j$  with the origin, i.e. the axis of symmetry through  $p_j$  of the pentagon formed by  $\{p_j\}_{j=0}^4$ . The corresponding self-covering is depicted in Figure 6.4. Here the equivalence classes are of the form  $\bar{k}S_k w$  where  $k \in X$  and  $S_k = \{k - 2 \pmod{5}, k + 2 \pmod{5}\}$ .

Our construction from Section 5 yields the p.c.f. group  $G_{\mathcal{L}}$  generated by

$$\begin{aligned} a_0 &= (23)(a_0, 1, 1, 1, 1), \\ a_1 &= (34)(1, a_1, 1, 1, 1), \\ a_2 &= (40)(1, 1, a_2, 1, 1), \\ a_3 &= (01)(1, 1, 1, a_3, 1), \\ a_4 &= (12)(1, 1, 1, 1, a_4), \end{aligned}$$

so that  $\mathcal{L}$  is a self-similar structure on the limit space of  $G_{\mathcal{L}}$ . The Moore diagram of the nucleus of  $G_{\mathcal{L}}$  is shown in Figure 6.5.

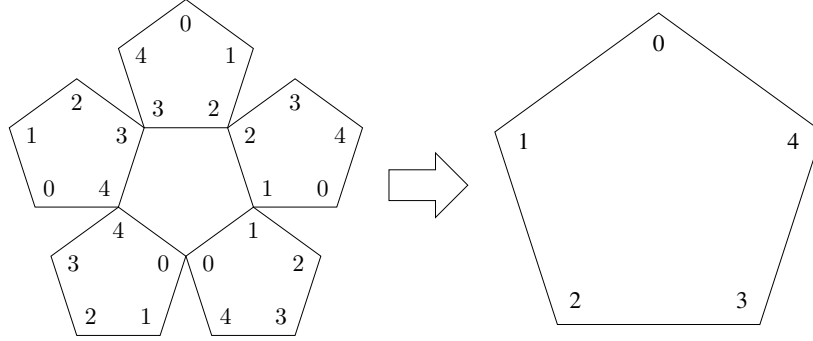
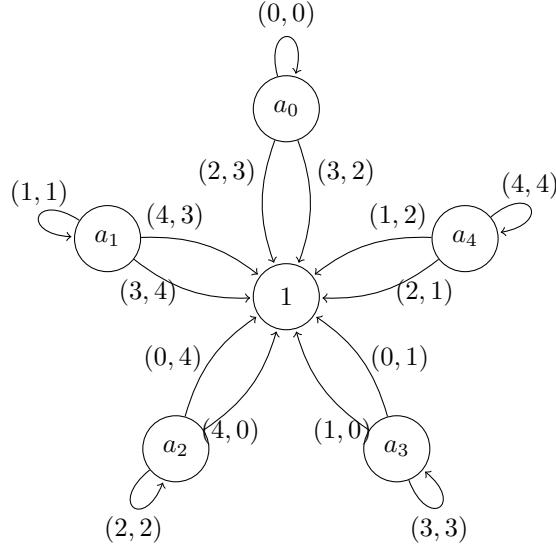


FIGURE 6.4. Self-covering of pentakun in Example 6.5

FIGURE 6.5. Nucleus of the group  $G_{\mathcal{L}}$  in Example 6.5

It is easy to perform an analogous construction for all  $n$ -kuns where  $n$  is odd.

**Example 6.6** (Hexakun and Linstrøm Snowflake). In Example 6.5 we showed how a self-covering could be constructed on the pentakun that could be taken to be the shift map required for the construction in Section 5. The current example shows the way to construct a self-covering for the hexakun, a fractal analogous to the pentakun but constructed instead from a hexagon. We shall also discuss why no self-covering can be constructed for the Linstrøm snowflake, a nested fractal which is a variation of the hexakun.

Similar to the pentakun, the hexakun is typically constructed (e.g. in [Kig01]) as the unique non-empty compact space  $K \subset \mathbb{C}$  invariant under the injections

$$f_k(z) = \frac{1}{3}(z - p_k) + p_k, \quad \text{where } p_k = e^{\pi i k/3}.$$

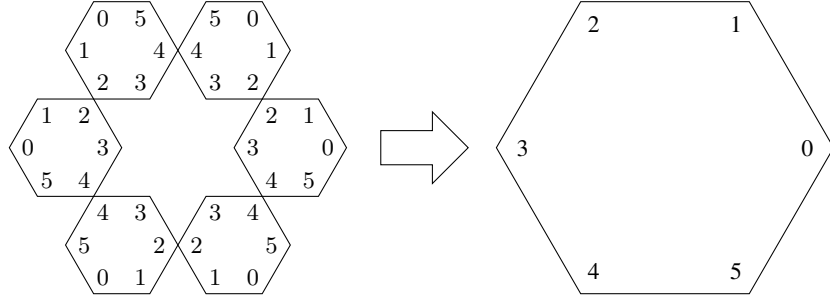


FIGURE 6.6. Self-covering of hexakun in Example 6.6

Writing  $X = \{0, 1, 2, 3, 4, 5\}$ , we see that  $\mathcal{L}_0 = (K, X, \{f_j\}_{j=0}^5)$  is the usual self-similar structure. As with the Sierpiński gasket and the pentakun, this self-similar structure does not admit a shift map, and so we have to choose another self-similar structure.

This fractal is a set of six copies of itself, each with a “corner” at the point  $p_k$ , and joined to two adjacent copies at the corner, two corners away from  $p_k$ . Our self-covering can be thought of as folding the fractal in half along the  $y$ -axis, so the cells in the left half-plain land on their reflections in the right. For the 3 cells on the right, we fold the upper and lower cells onto the cell containing  $p_0$ . Finally, we rescale the  $p_0$  cell using the map  $f_0^{-1}$ .

Formally, if we let  $\phi_1(z) = e^{\pi i/3}\bar{z}$ ,  $\phi_2(z) = e^{2\pi i/3}z$ ,  $\phi_3(z) = -\bar{z}$ ,  $\phi_4(z) = e^{4\pi i/3}z$ , and  $\phi_5(z) = e^{5\pi i/3}\bar{z}$  (here  $\bar{z}$  is the complex conjugate), then our modified self-similar structure is  $\mathcal{L} = (K, X, \{F_j\}_{j=0}^5)$  where  $F_0 = f_0$  and  $F_j = \phi_j \circ f_j$  for  $j = 1, 2, 3, 4, 5$ .

We now find the critical set  $\mathcal{C}$  and post-critical set  $\mathcal{P}$  of this new self-similar structure  $\mathcal{L}$ . Noticing that the fixed point of  $F_0$  is still  $p_0$ , we see that  $\pi(\bar{0}) = p_0$ . Also,  $F_j(p_0) = p_j$ , and so  $\pi(\bar{0}j) = p_j$  for  $j = 1, 2, 3, 4, 5$ . Since cells are only joined at the corners, this is enough to give us the addresses for the entire critical set; thus,  $\mathcal{C} = \{\bar{0}2j, \bar{0}4j \mid j \in X\}$ . Notice that in this self-similar structure, only the points  $p_2$  and  $p_4$  are mapped to boundary points. The post-critical set is then given by  $\mathcal{P} = \{\bar{0}, \bar{0}2, \bar{0}4\}$ .

More precisely, we examine the self-similar structure and write down the equivalence classes as follows:

$$\begin{aligned} \bar{0}20w &\sim \bar{0}21w, & \bar{0}22w &\sim \bar{0}23w, & \bar{0}24w &\sim \bar{0}25w, \\ \bar{0}40w &\sim \bar{0}45w, & \bar{0}41w &\sim \bar{0}42w, & \bar{0}43w &\sim \bar{0}44w. \end{aligned}$$

Applying our construction from Section 5, we get a group  $G_{\mathcal{L}}$  generated by the nine elements with wreath recursions

$$a_{01} = (01), \quad a_{23} = (23), \quad a_{45} = (45),$$

and

$$\begin{aligned} b_0 &= (b_0, 1, a_{01}, 1, 1, 1), & b_1 &= (b_1, 1, a_{23}, 1, 1, 1), & b_2 &= (b_2, 1, a_{45}, 1, 1, 1), \\ b_3 &= (b_3, 1, 1, 1, a_{01}, 1), & b_4 &= (b_4, 1, 1, 1, a_{23}, 1), & b_5 &= (b_5, 1, 1, 1, a_{45}, 1). \end{aligned}$$

Notice that according to our construction, there are six elements in the image  $\pi(\mathcal{C})$  of the critical set, and each of these corresponds to 2 generators, and so one may expect to obtain twelve generators. However, upon closer examination, we see that, for example,  $\pi(\bar{0}20) = \pi(\bar{0}21)$  and  $\pi(\bar{0}40) = \pi(\bar{0}41)$  both give rise to the generator with wreath recursion (01); thus, we see that three generators are redundant, and so  $G_{\mathcal{L}}$  is generated by nine elements.

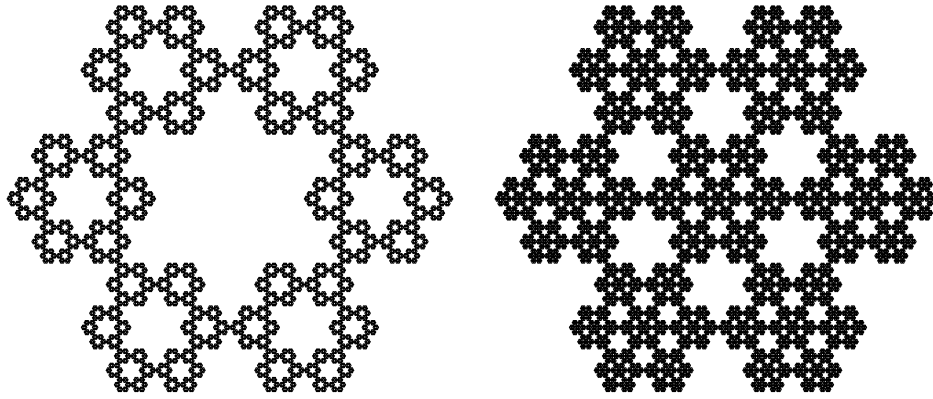


FIGURE 6.7. Hexakun and Lindstrøm snowflake

We now turn to the Linstrøm snowflake, which is a variation on the hexakun. It is the unique non-empty compact space  $L \subset \mathbb{C}$  invariant under the injections  $f_0, \dots, f_6$ , where  $f_j$  are the same as above for  $0 \leq j \leq 5$ , and  $f_6(z) = z/3$ . This fractal is like the hexakun, but with a scaled copy of itself inserted into the center. Cells of the snowflake still only intersect at the “corners;” in particular,  $f_j(L) \cap f_k(L) = \{f_j(p_n) \mid f_j(p_n) = f_k(p_m) \text{ for some } m\}$  contains at most one element.

Suppose now that there exists some self-similar structure  $\mathcal{L}' = (L, X, \{g_j\}_{j=0}^6)$  on  $L$  that has a shift map  $\sigma$ . Without loss of generality, we can assume that  $g_6(L)$  is the first-level scaled copy of  $L$  in the center. Notice that  $g_6(L)$  intersects every other cell at one point, so  $g_6(p_0) = g_j(p_0)$  for some  $j$ , and  $g_6(p_1) = g_k(p_1)$  for some other  $k$ . Then  $g_j(L)$  must intersect  $g_k(L)$  at one point  $p$ , such that  $g_j^{-1}(p) = g_k^{-1}(p) = \sigma(p)$  is a boundary point adjacent to both  $p_0$  and  $p_1$ . Since no such boundary point exists, we have arrived at a contradiction. Therefore, there exists no self-similar structure on  $L$  that admits a shift map, and so the Linstrøm snowflake cannot arise as the limit space of a contracting group.

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